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Renormalization of modular invariant Coulomb gas and sine–Gordon theories, and the quantum Hall flow diagram

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Abstract. Using the renormalization group (RG) we study two-dimensional electromagnetic Coulomb gas and extended sine–Gordon theories invariant under the modular group $SL(2,\mathbb{Z})$. The flow diagram is established from the scaling equations, and we derive the critical behaviour at the various transition points of the diagram. Following the proposal for a $SL(2,\mathbb{Z})$ duality between different quantum Hall fluids, we discuss the analogy between this flow and the global quantum Hall phase diagram.

1. Introduction

In statistical physics, self-duality in the sense of Kramers and Wannier maps the high-temperature regime of a given model with its low-temperature regime. In two dimensions, examples of self-dual theories are provided by the two (Ising), three and four state Potts, the Ashkin Teller, and the clock models. All of them can be represented in terms of an electromagnetic Coulomb gas (Kadanoff 1978, 1979), allowing one to understand this self-duality as an exchange of the electric and magnetic components of the charges of the equivalent Coulomb gas (Kadanoff 1978). This self-duality allows one to locate the transition point, whose study, however, requires a full renormalization group (RG) study, either directly on the Coulomb gas formulation (Nienhuis 1987) or on the associated sine–Gordon theory (Boyanovsky 1989).

More than 15 years ago, Cardy uncovered a generalization of this simple duality by adding a topological coupling between magnetic and electric charges of a two-flavour Coulomb gas (Cardy 1982). Motivated by the study of oblique confinement in four-dimensional $\mathbb{Z}_p$ lattice gauge theory with a topological θ term, Cardy and Rabinovici (1982) formulated a Coulomb gas where the presence of the θ term considerably enlarged the usual Kramers–Wannier duality $g \to 1/g$ to the full modular group $SL(2,\mathbb{Z})$ (Cardy 1982). This θ coupling was later extended to higher dimensions in Shapere and Wilczek (1989). In analogy to the work of Kramers and Wannier, Cardy (1982) derived the location of the numerous transition points in the presence of this topological coupling, as the invariant points of the modular group.

The purpose of this paper is to explicitly study using the RG, the scaling behaviour of an extension of the Cardy–Rabinovici model. Besides the critical behaviour associated with the transition points identified by Cardy, this renormalization study allows one to find the scaling flow of the model: the fixed points defining different phases are defined and related to the transitions points. Thus, we find a condition that two different phases must fulfil to be related by a transition. This is, to our knowledge, the first explicit renormalization of a two-dimensional modular invariant model.

Besides the applications of this extended Coulomb gas to the previously cited statistical models in two dimensions and to fermion models in 1 + 1 dimension via its sine–Gordon representation, such two-dimensional modular invariant theories are of interest for the study of the global phase diagram of the quantum Hall effect. This effect corresponds to the quantization of the transverse conductivity σ_{xy} of a two-dimensional gas of electrons in a strong transverse magnetic field, and it is now well understood in terms of the microscopic Laughlin wavefunctions (Laughlin 1983). The problem of the nature of the transitions between the different quantum Hall fluids is of particular interest and remains unsolved. In particular, the notion of superuniversality was proposed to explain the similar behaviour of all the transitions in the global phase diagram of the quantum Hall effect (Kivelson *et al* 1992). Such a superuniversality can be deduced from a duality of the underlying model, relating all the transition points with each other. Indeed, the similarity between the phase diagram of Cardy (1982) and the expected renormalization flow diagram in the two-parameter scaling model of the quantum Hall effect was noticed (Shapere and Wilczek 1989) before this proposal. Later on, Lütken and Ross (1992) more precisely related the properties of this phase diagram with the presence of a $SL(2, \mathbb{Z})$ symmetry, in the framework of a two-parameter σ_{xx}, σ_{xy} scaling theory.

On the other hand, the experiments of Shahar *et al* (1996) on the transition between the $\nu = \frac{1}{3}$ and the Hall insulator have shown a reflection relation between the nonlinear current density and electric field on both sides of the transition. Within the effective description of the quantum Hall effect, this reflection was interpreted as a duality which exchanges the electric and magnetic component of the low-lying excitations (Shahar *et al* 1996). Motivated by these experimental results, several authors focused on modular invariant models (Fradkin and Kivelson 1996, Pryadko and Zhang 1996). Even though no derivation of a microscopic modular invariant model exists, these studies focused on the general constraint from the modular symmetry on the phase diagram and expressions for the conductivity at the transitions (Lütken and Ross 1992, Dolan 1998). The phase diagram of the modular invariant Coulomb gas we study in this paper is expected to mimic the one of the quantum Hall effect. As an explicit renormalization of a modular invariant model in two dimensions, we expect this study to be helpful for the description of the critical behaviour of the quantum Hall transitions.

In another context, it is interesting to note that modular invariant Coulomb gas also appeared in the work of Callan and Freed (1992) on the dissipative motion of a charged particle in two dimensions in a transversed magnetic field and a periodic electric potential. Using a mapping to a one-dimensional version of the model of Cardy and Rabinovici (1982), these authors found a phase diagram as a function of the dissipation strength and the magnetic field which resembles the diagram obtained from our renormalization study.

This paper is organized as follows: in section 2.1 we define the model both in its sine–Gordon version and as a lattice Villain model with a topological θ term. The mapping to an electromagnetic Coulomb gas is then obtained in section 2.2 together with the symmetry $SL(2, \mathbb{Z})$ in section 2.3. In section 3.1 we derive the general RG equations for the model using a Kosterlitz scheme, and the phase diagram is established in section 3.3 together with the critical behaviour at the critical points of this diagram.

2. Electromagnetic Coulomb gas

2.1. Extended sine–Gordon model and theta terms

In this paper we consider the two-dimensional extended sine–Gordon model defined in its more general form by the partition function

$$Z = \prod_a \int \mathrm{d}[\phi^a]\, \mathrm{e}^{\mathcal{A}[\phi^a,\tilde{\phi}^a]}$$

with the action

$$\mathcal{A}[\phi^a,\tilde{\phi}^a] = \int \frac{\mathrm{d}^2\boldsymbol{r}}{a^2}\Bigg(-\frac{1}{4\pi}\sum_{a,b=1}^{2k}[g^{-1}]^{ab}\partial_i\phi^a\partial_i\phi^b + 2y\sum_a \cos(\phi^a) + 2\tilde{y}\sum_a \cos p([g^{-1}]^{ab}\tilde{\phi}^b - M^{ab}\phi^b)\Bigg). \tag{1}$$

In this definition we consider $2k$ fields ϕ^a and their dual fields $\tilde{\phi}^a$ defined by $i\partial_i\tilde{\phi} = \epsilon_{ij}\partial_j\phi$, where $i = 1, 2$ labels the two directions of the plane, and ϵ_{ij} is the antisymmetric tensor. The coupling g^{ab} and M^{ab} are $2k \times 2k$ matrices; M^{ab} has to be antisymmetric for renormalizability of the model (see the following): $M^{ab} = -M^{ba}$. In this context the operators

$$\mathcal{O}_{\boldsymbol{n},\boldsymbol{m}}(\boldsymbol{r}) = \exp \mathrm{i}\Bigg(\sum_a n_a\phi^a(\boldsymbol{r}) + \sum_a m_a\tilde{\phi}^a(\boldsymbol{r})\Bigg) \tag{2}$$

correspond to the parafermions operators of Fradkin and Kadanoff (1980) which create a vector electric charge $\boldsymbol{n}$ and magnetic charge $\boldsymbol{m}$ in site $\boldsymbol{r}$.

Alternatively, we can consider a $\mathbb{Z}_p$ gauge theory defined on the square lattice by $2k$ Villain models perturbed by symmetry breaking fields of strength y and coupled by a (topological) term:

$$\mathcal{A}_{Villain} = -\frac{1}{4\pi}\sum_\alpha\sum_{i=1,2}[g^{-1}]^{ab}(\partial_i\phi^a_\alpha - 2\pi pA^a_{\alpha,i})(\partial_i\phi^b_\alpha - 2\pi pA^b_{\alpha,i}) + \ln(y)\sum_\alpha n^a_\alpha n^a_\alpha + \mathrm{i}\sum_\alpha n^a_\alpha\phi^a_\alpha - \mathrm{i}\frac{1}{4\pi}\sum_\alpha M^{ab}\epsilon_{ij}(\partial_i\phi^a_\alpha - 2\pi pA^a_{\alpha,i})(\partial_j\phi^b_\alpha - 2\pi pA^b_{\alpha,j}) \tag{3}$$

where the $A^a_{\alpha,i}$ correspond to the integer-valued gauge field defined on the bonds of the square lattice, and $\partial_x\phi^a_\alpha$ is the discrete derivative $\phi^a_{\alpha+x} - \phi^a_\alpha$. Here, and in the following, summation over repeated indices is assumed.

The meaning of the coupling between the different fields ϕ^a becomes clearer upon restriction to a two-component model ($k = 1$) with the coupling constants:

$$g^{ac} = g\delta^{ac} \qquad M^{ac} = \frac{\theta}{2\pi}\epsilon^{ac} \qquad (k = 1). \tag{4}$$

By considering the four-component field $\phi_\mu = (0, 0, \phi^1, \phi^2)$ which only depends on x_1, x_2 we can write the quadratic term in (3) as $F_{\mu\nu}F^{\mu\nu}$ where $\mu, \nu = 1, \ldots, 4$ and $F_{\mu\nu}$ is the usual electromagnetic tensor associated with the field ϕ^μ: $F_{\mu\nu} = \partial_\mu\phi_\nu - \partial_\nu\phi_\mu - 2\pi pA_{\mu\nu}$. The second term can be interpreted as a coupling between the electric particle and the field ϕ_μ, while the last term reads $(\theta/2\pi)\epsilon_{\mu\nu\rho\sigma}F^{\mu\nu}F^{\rho\sigma} = (\theta/2\pi)F^{\mu\nu}\tilde{F}_{\mu\nu}$, which is known as a topological θ term. This model and its Coulomb gas formulation in two dimensions was first studied by Cardy and Rabinovici (1982). The general M coupling in (1) is thus the analogue of this topological coupling for the $2k$ real component field in two dimensions. Other extensions to higher dimensions of space were developed by Shapere and Wilczek (1989).

Although in the following, we will derive the RG equation for the general model (1), we only analyse in detail the scaling behaviour of the simpler model (4) with electric charges in the first component ($a = 1$) and magnetic in the second (see below). This model is defined by the following restriction of (1):

$$\mathcal{A} = \frac{-1}{4\pi g} \int \frac{d^2\boldsymbol{r}}{a^2}((\partial\phi^1)^2 + (\partial\phi^2)^2) + y\cos(\phi^1) + \tilde{y}\cos p\left(g^{-1}\tilde{\phi}^2 - \frac{\theta}{2\pi}\phi^1\right). \tag{5}$$

We can now express the partition function of the above models in terms of electromagnetic Coulomb gas, extending the usual case of Nienhuis (1987).

2.2. $SL(2, \mathbb{Z})$ invariant electromagnetic Coulomb gas

To obtain the Coulomb gas formulation of the above model, we first use the Villain approximation (Villain 1975) of the cosine coupling in (1):

$$e^{2y\cos\phi} \sim \sum_{n=0,\pm1} e^{in\cdot\phi + n^2\ln(y)}.$$

This approximation is valid for small coupling strength y. The models with both forms of the interaction are known to be in the same universality class without the topological term (Boyanovsky 1989), which ensures the same in our case. Within this approximation, the partition sum of (1) consists of a trace over the fields $\phi^a(\boldsymbol{r})$ and the electric and magnetic charge density $n^a(\boldsymbol{r})$ and $\boldsymbol{m}^a(\boldsymbol{r})$ of the exponential of the action

$$\tilde{\mathcal{A}} = \int \frac{d^2\boldsymbol{r}}{a^2}\bigg(-\frac{1}{4\pi}\sum_{a,b}[g^{-1}]^{ab}\partial_i\phi^a\partial_i\phi^b + i\sum_a \phi^a(n^a + pM^{ab}m^b) + ip\sum_a \tilde{\phi}_a[g^{-1}]^{ab}m_b$$
$$+ \ln(y)\sum_a (n^a)^2 + \ln\tilde{y}\sum_a (m^a)^2\bigg).$$

In this bare model the vector charges $\boldsymbol{n}(\boldsymbol{r})$ have components $0, \pm1$, however, upon coarse-graining, charges with higher components will be generated. The lattice Villain model with a topological coupling (3) leads exactly to the same sum, where the magnetic charges $m^a(\boldsymbol{r})$ are located on the sites of the dual lattice. They are defined by the oriented sum of the potential $\boldsymbol{A}^a$ over the plaquette surrounding the dual site: $m^a(\boldsymbol{r}) = -\epsilon_{ij}\partial_i A^a_j$.

After integration over the field $\phi^a(\boldsymbol{r})$, and using the neutrality of the charges $\int_{\boldsymbol{r}} n^a(\boldsymbol{r}) = \int_{\boldsymbol{r}} m^a(\boldsymbol{r}) = 0$ (imposed by the infrared regularization), we obtain an electromagnetic two-dimensional Coulomb gas of electric n^a and magnetic m^a charges, defined either on a lattice (and its dual for the m charge) or in the continuum, which take value in $\mathbb{Z}^{2k}$. It is defined by the grand canonical partition function

$$Z = \sum_{[\boldsymbol{n}_i,\boldsymbol{m}_i]}' \left(\prod_{i=1}^N \int \frac{d^2\boldsymbol{r}_i}{a^2} y_{\boldsymbol{n}_i,\boldsymbol{m}_i}\right) e^{\mathcal{A}_{cg}[\boldsymbol{n},\boldsymbol{m}]} \tag{6}$$

where the $y_{\boldsymbol{n},\boldsymbol{m}}$ are the charge fugacities, and the primed sum counts each distinct configuration only once. The corresponding action can be written in its most general form as

$$\mathcal{A}_{cg}[\boldsymbol{n},\boldsymbol{m}] = \tfrac{1}{2}[g^{ac}(n^a + pM^{ab}m^b) * G * (n^c + pM^{cd}m^d) + p^2(g^{-1})^{ac}m^a * G * m^c]$$
$$-ipn^a * \Phi * m^b \tag{7}$$

where we assumed summation over repeated indices and used the contraction notation: $n * G * n = \sum_{i\neq j} n_i G(\boldsymbol{r}_i - \boldsymbol{r}_j) n_j$. In this action, the potentials G and Φ are defined respectively by the propagators

$$g^{ab}G(r) = \langle\phi^a(\mathbf{0})\phi^b(\mathbf{0})\rangle - \langle\phi^a(\mathbf{0})\phi^b(\boldsymbol{r})\rangle$$
$$-i\Phi(r) = \langle\phi^a(\mathbf{0})\tilde{\phi}^b(\mathbf{0})\rangle - \langle\phi^a(\mathbf{0})\tilde{\phi}^b(\boldsymbol{r})\rangle.$$

In a neutral Coulomb gas, these propagators only need to be regularized at short distances: in the following we will use a real-space hard cut-off a, corresponding to hard core charges. The asymptotics of these interactions is given by $G(\boldsymbol{r}) + \mathrm{i}\Phi(\boldsymbol{r}) \sim_{r\gg a} \ln(x + \mathrm{i}y)/a$. For a definition of these potentials on the lattice, see Kadanoff (1979, appendix A).

2.3. Symmetries for two components charges ($k = 1$)

We now consider more precisely the model (5). With the presence of the θ (or M) coupling, the usual Kramers–Wannier duality $g \leftrightarrow p^2 g^{-1}$ of the electromagnetic Coulomb gas (Kadanoff 1978) is considerably enlarged (Cardy 1982). Besides the time reversal symmetry $TS[\{n^a\}, \{m^a\}, g, \theta] = S[-\{n^a\}, \{m^a\}, g, -\theta]$, the 2π periodicity of the θ coupling translates into $PS[\{n^a\}, \{m^a\}, g, \theta] = S[\{n^a - \epsilon^{ab} \cdot m^b\}, \{m^b\}, g, \theta + 2\pi]$. Finally, the action (7) is also invariant under the self-dual transformation $D\ S[\{n^a\}, \{m^a\}, g, \theta] = S[\{\epsilon^{ab} \cdot m^b\}, \{\epsilon^{ab} \cdot n^b\}, g', \theta']$ with

$$\frac{g}{p^2} = \frac{1}{g'} + g'(\theta')^2 \qquad g\theta = -g'\theta'. \tag{8}$$

These transformations are better parametrized using the complex coupling constant $z = p\frac{\theta}{2\pi} + \mathrm{i}pg^{-1}$: the above transformation simply reads $P(z) = z + 1$ and $D(z) = -z^{-1}$. From this it is obvious D and P do not commute, and they generate the whole infinite discrete group $SL(2, Z)$. The purpose of this paper is thus to explicitly study the behaviour of the modular invariant Coulomb gas (7) under the RG.

3. Renormalization à la Kosterlitz

3.1. Renormalization group equations

Without a θ (M) term, the electromagnetic Coulomb gas (7) can be renormalized either by following the (Anderson–Yuval) Kosterlitz scheme (Nienhuis 1987) or directly by an operator product expansion in the sine–Gordon formulation (Boyanovsky 1989). Both methods use the product of the parafermion operators (2) and give the same results. Hence, for the sake of simplicity, we will follow the Kosterlitz approach, extending the classical method to the presence of the θ term (see Nienhuis (1987) for a detailed review of this method).

Upon coarse-graining the model by increasing the hard core cut-off a, we leave the partition function (6) invariant by defining scale-dependent coupling constants and fugacities. Three different contributions to these variables have to be considered: naive rescaling, fusion and annihilation of electromagnetic charges. The naive rescaling comes simply from the change of cut-off in the integration measure and the interaction $G(r)$: it gives the eigenvalue of the fugacities $y_{\boldsymbol{n},\boldsymbol{m}}$. Upon infinitesimal increase of the cut-off $a \to \tilde{a} = a\mathrm{e}^{dl}$, the distance between two neighbouring charges can become less than the new cut-off $\tilde{a}$. When these two charges form a dipole, we integrate them out (annihilation of charges), while we simply glue them into a single charge otherwise (fusion of charges). In the latter case we get a correction to the fugacity of the new charge, which together with the naive rescaling, gives the following scaling equation for the fugacities:

$$\partial_l y_{\boldsymbol{n},\boldsymbol{m}} = \left(2 - \frac{g}{2}\sum_a (n^a + pM^{ab} \cdot m^b)^2 - \frac{p^2}{2g}\sum_a (m^a)^2\right) y_{\boldsymbol{n},\boldsymbol{m}} + \pi \sum_{(\boldsymbol{n}',\boldsymbol{m}')+(\boldsymbol{n}'',\boldsymbol{m}'')=(\boldsymbol{n},\boldsymbol{m})} \delta_{\boldsymbol{n}'\cdot\boldsymbol{m}''+\boldsymbol{n}''\cdot\boldsymbol{m}'}\, y_{\boldsymbol{n}',\boldsymbol{m}'}\, y_{\boldsymbol{n}'',\boldsymbol{m}''} \tag{9a}$$

with the notation $\boldsymbol{n} \cdot \boldsymbol{m} = \sum_a n^a m^a$.

In the other case, the small dipoles we annihilate upon coarse-graining screen the interaction between distant charges. In the partition function (6), the term involving two (nonzero) charges $(\boldsymbol{n}_s, \boldsymbol{m}_t)$ and $(\boldsymbol{n}_s, \boldsymbol{m}_t)$ distant from $a < |\boldsymbol{r}_s - \boldsymbol{r}_t| < a\mathrm{e}^{dl}$ can be expanded in $a/|\boldsymbol{r}_i - \boldsymbol{r}_j|$ (the distance between distant charges) and yields

$$-2\pi^2\,\mathrm{d}l\int_{|\boldsymbol{r}_i-\boldsymbol{r}_j|>\tilde{a}}\sum_{(\boldsymbol{n}_i,\boldsymbol{m}_i);(\boldsymbol{n}_j,\boldsymbol{m}_j)} y_{\boldsymbol{n}_i,\boldsymbol{m}_i} y_{\boldsymbol{n}_j,\boldsymbol{m}_j}(\alpha_{is}\alpha_{js} - \beta_{is}\beta_{js})y^2_{\boldsymbol{n}_s,\boldsymbol{m}_s} G(\boldsymbol{r}_i - \boldsymbol{r}_j)$$

where $\alpha_{js} = g^{ac}N_j^a N_s^c + p^2(g^{-1})^{ac}m_j^a m_s^c$, $\beta_{js} = p(n_j^a m_s^a + n_s^a m_j^a)$ and we have defined the composite electric component of the charges $N_j^a = n_j^a + pM^{ab}m_j^b$. Using the antisymmetry of M^{ab} and re-exponentiating this contribution, we can check the renormalizability of the model to order y^2 as this contribution can be cast into a contribution to the matrices g, M and the fugacities. We obtain the following corrections to g and M to the order y^2:

$$\partial_l g^{ac} = -2\pi^2 \sum_{\boldsymbol{n}_s,\boldsymbol{m}_s}(g^{ab}g^{cd}N_s^b N_s^d - p^2 m_s^a m_s^c)y^2_{\boldsymbol{n}_s,\boldsymbol{m}_s} \tag{9b}$$

$$\partial_l M^{ab} = -2\pi^2 p \sum_{\boldsymbol{n}_s,\boldsymbol{m}_s}[(g^{-1})^{bc}N_s^a m_s^c - (g^{-1})^{ac}N_s^b m_s^c]y^2_{\boldsymbol{n}_s,\boldsymbol{m}_s}. \tag{9c}$$

Notice the antisymmetry of the correction (9*c*) to M^{ab}. We can now derive the renormalization flow from these scaling equations.

3.2. *Specific model and charge asymmetry*

In the following we will restrict our study to the model (5). In contrast to the usual case where the fugacities of electromagnetic charges are symmetric both in n and m (Nienhuis 1987), here this symmetry is broken by the presence of the θ coupling. As seen above with the T symmetry, changing n to $-n$ amounts to also changing θ to $-\theta$, and similarly with the transformation $m \to -m$. The only symmetry which does not modify either g or θ corresponds to $(n, m) \to (-n, -m)$, which ensures the neutrality of the gas for any value of the couplings. Hence, we cannot assume the m or n parity of the fugacities $y_{n,m}$ as the condition $y_{n,m} = y_{-n,m} = y_{n,-m}$ will not be preserved by the RG.

For the model (5) we need only consider charges satisfying $n^2 = 0$, $m^1 = 0$. We will use the notation (n, m) for (n^1, m^2). The above RG equations (9) can then be written in a more pleasant way, using the notation $x = pg^{-1}$ and $t = p\theta/2\pi$:

$$\partial_l x = 2\pi^2 p \sum_{n,m}[(n + tm)^2 - x^2m^2]y^2_{n,m} + \mathcal{O}(y^3) \tag{10a}$$

$$\partial_l t = -4\pi^2 px \sum_{n,m} m(n + tm)y^2_{n,m} + \mathcal{O}(y^3) \tag{10b}$$

$$\partial_l y_{n,m} = 2\left(1 - \frac{p}{4x}(n + tm)^2 - \frac{p}{4}xm^2\right)y_{n,m} + \mathcal{O}(y^2). \tag{10c}$$

In the last equation the nonlinear term from (9*c*) has been omitted, being subdominant around each transition point. However, it has to be taken into account when analysing the topology of the whole phase diagram. The above RG equations correspond to the starting point of our analysis of the different transitions. Their invariance under the duality transformations is made more explicit if we write them as a single equation for the complex coupling constant z:

$$\partial_l z = \mathrm{i}2\pi^2 p \sum_{n,m}(n + zm)^2 y^2_{n,m}. \tag{11}$$

In this last equation, the invariance under duality of the flow $(z \to -z^{-1})$ is obvious.

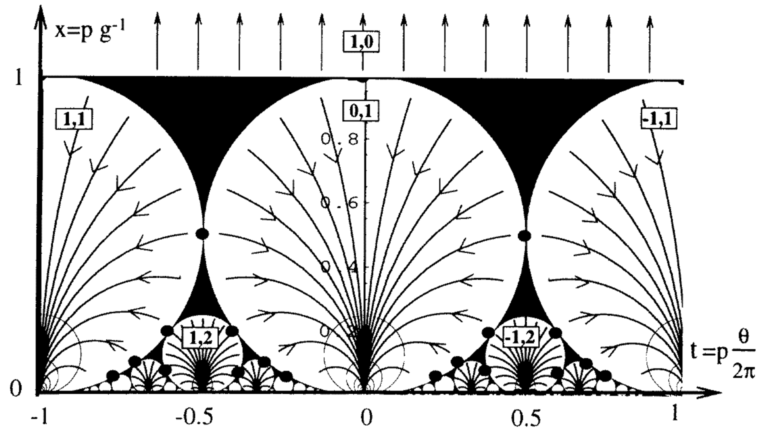

Figure 1. RG flow in the plane $p\frac{\theta}{2\pi}, pg^{-1}$. The flow are shown up to $m = 3$. The regions inside the circle correspond to stability regions for charges (n, m) with nonzero n, m labelling the size of the circle while n gives their positions. For $pg^{-1} > 1$ only charges with $m = 0$ exist. The transition points between $m \neq 0$ phases are shown as black points while the black region corresponds to a phase with no charge (neutral phase).

The invariance under duality of these beta functions, which was assumed in previous studies of modular invariant models, have important consequences: as the axis $t = 0$ is invariant (in the limit of vanishing fugacities) and thus correspond to a flow line of the above equations, we know from the action of the modular group on this flow line that all the flow lines of the phase diagram are either straight lines or arcs of circles. The flow diagram obtained by numerically integrating these equations is shown in figure 1, where only charges with $m \leqslant 3$ have been taken into account.

3.3. Analysis of the flow

To derive the phase diagram from the renormalization analysis, we first find the domains where a charge (n, m) proliferates. Such a charge proliferation, or equivalently the parafermion operator (2) associated with a charge $(n + \frac{\theta}{2\pi}m, m)$, is relevant when the associated fugacity increases under rescaling. From the renormalization eigenvalue of the fugacities in (10*c*) we deduce that the pure electric charges $(n, 0)$ are only relevant for $x > pn^2/4$. For $x < 1$ the composite charges (n, m) become relevant inside the circles $[n, m]$ centred on $(t, x) = (-n/m, 2/(pm^2))$ of radius $2/(pm^2)$. All these circles are thus tangent to the axis $x = 0$ (see figure 1).

Each circle $[n, m]$ is characterized by a ratio $\nu = n/m$ instead of n, m, as all charges $(\lambda n, \lambda m)$ with $\lambda \in \mathbb{Z}$ are generated under renormalization in this circle (see figure 1). In the following we will write $\nu = n/m$ where m is the minimal magnetic charge allowed in the circle. Under renormalization, in a circle $\nu = n/m$, $x = pg^{-1}$ flows to zero while t is renormalized to $t^* = \nu$. Hence each rational point of the axis $x = 0$ corresponds to a fixed point of the RG, characterizing a given phase. The phase $m = 0$, stable for $x > pn^2/4$, can be viewed as a special circle of infinite radius, characterized by $\nu = \infty$. In this phase $x \to \infty$ and t is slightly renormalized to a (non-universal) real value. Below $x < pn^2/4$ and between all these circles, all the fugacities renormalize to zero, which corresponds to a neutral phase of the Coulomb gas, characterized by non-universal renormalized g and θ and vanishing fugacities.

Transitions in this phase diagram are *a priori* of two different types: the transitions between two circles characterized by a different ratio ν and the transition between a phase with relevant charge (n, m) and the neutral phase. Transitions of the first kind correspond to tangent points between circles in the phase diagram. Such points exist only when $p = 4$. For $p < 4$, the circles overlap and these transitions are no longer accessible by the present perturbative study, while for $p > 4$ all circles with different ratio ν are disconnected and these transitions disappear. In the following we will consider in more detail the case $p = 4$, whose phase diagram is shown on figure 1.

A transition between two phases (ratios) ν_1 and ν_2 is allowed if the two corresponding circles are at a tangent to each other, which can be written

$$\nu_1 - \nu_2 = \pm \frac{1}{m_1 m_2}. \tag{12}$$

The corresponding transition point is located in $(t, x) = (-(n_1 m_1 + n_2 m_2)/(m_1^2 + m_2^2), 1/(m_1^2 + m_2^2))$. To further analyse these transitions, we need to point out that any circle ν can be deduced from the circle $\nu = 0$ by succesive application of the transformations P, P^{-1} and the duality D. Moreover, the circle $\nu = 0$ itself is the image under D of the line of phase transitions $x = 1$. The transition points between the phase $\nu = 0$ and other $\nu' = \pm 1/m$ (see figure 1) are all images under D of the transition points between $[1, 0]$ and the phases $[n, 1]$, which are themselves related to the transitions $[1, 0] \leftrightarrow [0, 1]$ by the action of P and P^{-1}. Thus all the transition points between two ratios ν and ν' correspond to the same critical behaviour. Similarly all transitions between a phase ν and the neutral phase (no charge condensate) are images of the transition between $[1, 0]$ and this phase. We can now study these two transitions perturbatively in the $y_{n,m}$.

First, at a transition between $\nu_1 = n_1/m_1$ and $\nu_2 = n_2/m_2$ we can write renormalization equations for the distance orthonormal to the transition point δ_{12} while the distance perpendicular to this axis is not renormalized to lowest order. Using $\epsilon = (m_1^2 + m_2^2)\delta_{12}$ and $y = y_{n_1 m_1}$, $\tilde{y} = y_{n_2 m_2}$ we obtain

$$\partial_l \epsilon = 2\pi^2 (y^2 - \tilde{y}^2) \qquad \partial_l y = 2\epsilon y \qquad \partial_l \tilde{y} = 2\epsilon \tilde{y}$$

which correspond to a Kosterlitz–Thouless-like diverging correlation length $\xi \sim \exp(cte/|\epsilon|^{\frac{1}{2}})$. In both phases, the correlation functions of the parafermion operators decays exponentially:

$$\langle e^{i(n\phi(r) + m\tilde{\phi}(r))} e^{-i(n\phi(\mathbf{0}) + m\tilde{\phi}(\mathbf{0}))} \rangle \sim e^{-\frac{r}{\xi}}.$$

The transition between the neutral phase and a circle ν is describe by a critical point with one marginal direction (tangent to the circle) and the same diverging correlation length. While the correlation function have the same kind of exponential decays from the circle side of the transition, they decay algebraically in the neutral phase:

$$\langle e^{i(n\phi(r) + m\tilde{\phi}(r))} e^{-i(n\phi(\mathbf{0}) + m\tilde{\phi}(\mathbf{0}))} \rangle \sim \left(\frac{a}{r}\right)^{\frac{1}{x}(n+tm)^2 + xm^2}.$$

4. Analogy with the quantum Hall effect

Within the context of the two-parameter scaling theory of the quantum Hall effect, the first idea consists in identifying the coupling of the Coulomb gas model with the components of the conductivity tensor of the electron gas $\sigma_{xy} = t$, $\sigma_{xx} = x$. The ratio ν defined in the above study naturally translates into the filling factors of the quantum Hall states: each circle of the flow diagram corresponds to a given quantum Hall fluid and the renormalization equations we obtain provide the expected asymptotic value for this conductivity: $\sigma_{xy}^* = \nu$, $\sigma_{xx}^* = 0$. Following this analogy we can find that transitions between two filling factors ν_1, ν_2 are allowed if they satisfy the expected relation (12). All the allowed transitions between the two plateaux are in the same universality, being related by the duality D (8) which exchanges the electric and magnetic charges, as in Shahar *et al* (1996), thus supporting the superuniversality hypothesis (Kivelson *et al* 1992).

However, we notice that in our flow diagram, filling factors with any denominators exist, while only even denominators are expected for quantum Hall fluids. Moreover, fixed points corresponding to filling factors (circles) with even and odd denominators are related by modular transformations: hence one cannot avoid the even denominators in this model. This result has to be related to recent papers (Georgelin *et al* 1997) which reveal that the symmetry hidden behind the quantum Hall hierarchies should be the subgroup $\Gamma(2)$ of $SL(2,\mathbb{Z})$ instead of the whole modular group. This subgroup is generated by the transformation $z \to -z/(2z+1)$ (instead of $z \to -z^{-1}$ in this study) which preserves the oddness of filling factors. This transformation, when restricted to the axis $t = 0$ (imaginary axis), corresponds to $g \to g/(1+4g^2)$. Hence it does not exchange the high- and low-temperature phases of the usual electromagnetic Coulomb gas (Kadanoff 1978). Thus an extension of the usual electromagnetic model to a $\Gamma(2)$ invariant theory will not simply correspond to the addition of a topological term as for the model studied in this paper.

To conclude, let us comment in comparison with recent studies which focus on the constraint on the beta function from the modular symmetry (Dolan 1998). That study (together with Burgess and Lutken (1997)) assumes a modular (or $\Gamma(2)$) symmetry in a model with two parameters σ_{xx}, σ_{xy}, and derives the general form of the beta function of this model. In our work the model (6) possesses an infinite number of coupling constants corresponding to the electromagnetic charge fugacities. Thus a direct comparison between the beta functions is not possible. However, it should be noted that our renormalization procedure provides the first explicit example of the commutation of the modular symmetry and the renormalization, which is the central hypothesis of Burgess and Lutken (1997) and Dolan (1998).

5. Conclusion

In this paper we thus extended the renormalization study of Nienhuis (1987) to the Coulomb gas with a topological θ term in two dimensions. The renormalization scheme we used provides scaling equations which were themselves found to be invariant under the modular group. To our knowledge this study provides the first example in two dimensions of an explicit renormalization of a modular invariant model. The rich phase diagram of this model certainly deserves more work: of particular interest will be the applications of this renormalization study to the extensions of the various statistical models, related to the electromagnetic Coulomb gas in the absence of the θ term.

Note added in proof. After completion of this paper, a preprint (Cristofano, Giuliano and Nicodemi 1998) appeared on a related quantum Hall model where a finite electromagnetic background charge is added to the $k = 1$ restriction of the model (7). Although these authors used a simplified scheme, their scaling analysis seems to agree with the complete RG equations we derived in this paper.

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